

Rational Functions

Poles, Behavior at Infinity, Order, and the Singular Part at Infinity

Lecture Notes

September 24, 2026

1 Poles and Zeros

Definition (rational function)

$R(z) = \frac{P(z)}{Q(z)}$, where P, Q are polynomials with no common zeros (any common zeros can be factored out and cancelled).

$$R'(z) = \frac{P'(z)Q(z) - Q'(z)P(z)}{Q(z)^2}$$

exists wherever z is not a pole.

Definition (zeros of R)

The zeros of $P(z)$ are called the **zeros** of $R(z)$. The order (multiplicity) of a zero equals the order of the corresponding zero of $P(z)$.

Definition (poles of R)

The zeros of $Q(z)$ are called the **poles** of $R(z)$. The order (multiplicity) of a pole equals the order of the corresponding zero of $Q(z)$.

Example. $R(z) = \frac{(z-i)^2(z+1)}{(z+i)^3}$ has a pole at $-i$ of order 3, a zero at -1 of order 1, and a zero at i of order 2.

Remark. If z_0 is a pole of R , then $\lim_{z \rightarrow z_0} R(z) = \infty$ (in the spherical metric), so we set $R(z_0) := \infty$.

2 Behavior at Infinity

Consider $R_1(z) := R(1/z) = \frac{P(1/z)}{Q(1/z)}$ (the “ $1/z$ trick”), and define

$$R(\infty) := R_1(0) = \lim_{z \rightarrow \infty} R(z) = \lim_{z \rightarrow 0} R_1(z).$$

More details. If $P(z) = \sum_{k=0}^n a_k z^k$ and $Q(z) = \sum_{k=0}^m \ell_k z^k$ ($n = \deg P$, $m = \deg Q$, $a_n, \ell_m \neq 0$), then

$$R_1(z) = z^{m-n} \frac{\sum_{k=0}^n a_k z^{n-k}}{\sum_{k=0}^m \ell_k z^{m-k}} =: z^{m-n} R_0(z), \quad R_0(0) = \frac{a_n}{\ell_m} \neq 0, \infty.$$

So

$$R(\infty) = R_1(0) = \begin{cases} \frac{a_n}{\ell_m}, & \deg P = \deg Q \\ 0, & \deg P < \deg Q \\ \infty, & \deg P > \deg Q. \end{cases}$$

- If $\deg P < \deg Q$: ∞ is a zero of $R(z)$, of order $\deg Q - \deg P$ (the order of 0 as a zero of $R_1(z)$, since $R_1(z) = z^{m-n}R_0(z)$ with $R_0(0) \neq 0$).
- If $\deg P > \deg Q$: ∞ is a pole of $R(z)$, of order $\deg P - \deg Q$.
- If $\deg P = \deg Q$: ∞ is neither a pole nor a zero.

Total Count of Zeros and Poles

Suppose $\deg P > \deg Q$. The total number of zeros, counting order, is

$$\#\{\text{zeros of } P\} + \#\{\text{zeros at } \infty\} = \deg P + 0 = \deg P,$$

since ∞ is a pole, not a zero, in this case; and the total number of poles is

$$\#\{\text{zeros of } Q\} + \#\{\text{poles at } \infty\} = \deg Q + (\deg P - \deg Q) = \deg P.$$

Both totals equal $\deg P = \max(\deg P, \deg Q)$.

The case $\deg P < \deg Q$ follows by the same computation applied to $1/R = Q/P$, which swaps the roles of P, Q (and hence of zeros and poles): both totals equal $\deg Q = \max(\deg P, \deg Q)$. If $\deg P = \deg Q =: d$, then ∞ is neither a zero nor a pole, so both totals reduce to the finite count alone – $\#\{\text{zeros of } P\} = d$ and $\#\{\text{zeros of } Q\} = d$ – again equal to $\max(\deg P, \deg Q) = d$.

Consequence

The total number of zeros of R equals the total number of poles of R , equal to $\max(\deg P, \deg Q)$.

Definition (order of a rational function)

$\max(\deg P, \deg Q)$ is called the **order** of $R(z)$.

Remark. For every $w \in \mathbb{C}$ with $w \neq R(\infty)$, the equation $R(z) = w$ has $\deg R$ *finite* roots, counting multiplicity, since $R(z) - w$ has the same poles as $R(z)$ (as w is finite), hence the same order, and by the Consequence above its zeros – the solutions of $R(z) = w$ – number $\deg R$ in total. (If $w = R(\infty)$, one of these $\deg R$ roots, counted on the Riemann sphere, is $z = \infty$ itself, so the number of *finite* solutions is then $\deg R$ minus the multiplicity of ∞ as a root.)

Example 1: the Möbius (linear fractional) maps, $\frac{az+b}{cz+d}$, $ad-bc \neq 0$, $a, b, c, d \in \mathbb{C}$. Polynomials are exactly the rational functions with all poles at ∞ .

3 Partial Fraction Decomposition

Singular Part at Infinity

If $R(z) = P(z)/Q(z)$ with $\deg P > \deg Q$, polynomial division gives

$$P(z) = G_\infty(z)Q(z) + S(z), \quad \deg S < \deg Q,$$

so

$$R(z) = G_\infty(z) + \frac{S(z)}{Q(z)}, \quad H(z) := \frac{S(z)}{Q(z)}, \quad H(\infty) = 0.$$

G_∞ is a polynomial with $\deg G_\infty = \deg P - \deg Q > 0$ and $G_\infty(\infty) = \infty$: this is the **singular part of R at ∞** . If $\deg P < \deg Q$, set $G_\infty(z) \equiv 0$; if $\deg P = \deg Q$, set $G_\infty(z) \equiv R(\infty)$, a constant.

Observation. $\deg G_\infty = \max(\deg P - \deg Q, 0) =$ order of ∞ as a pole of R .